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MATHEMATICS

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AIM POINTS
MATHEMATICS
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XIth, XIIth, TARGET IIT-JEE (MAIN + ADVANCE) & COMPETITIVE EXAM. FOR XII (PQRS)

HIGHER ORDER DERIVATIVES

& Their Properties

CONTENTS

THINGS TO REMEMBER

1. If $y = f(x)$, then $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is called second order derivative of y with respect to x and is denoted by $\frac{d^2y}{dx^2}$ or, y_2 or y''.

   Similarly, third and higher order derivatives are defined.

2. If $x = f(t)$ and $y = g(t)$, then

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left\{\frac{g'(g)}{f'(g)}\right\}$$

or, $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left\{\frac{g'(g)}{f'(g)}\right\}\frac{d}{dx}$$

or, $$\frac{d^2y}{dx^2} = \frac{f'(t)g''(t) - g'(t)f''(t)}{\{f'(g)\}^3}$$

EXERCISE-1

1. If $y = A \cos nx + B \sin nx$, show that $\frac{d^2y}{dx^2} + n^2y = 0$.

2. If $y = Ae^{mx} + Be^{nx}$, show that $\frac{d^2y}{dx^2} - (m + n)\frac{dy}{dx} + mny = 0$

3. If $y = A \cos(\log x) + b \sin(\log x)$, prove that $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$.

4. If $y = \tan x + \sec x$, prove that $\frac{d^2y}{dx^2} = \frac{\cos x}{(1 - \sin x)^2}$

5. If $y = x \log\left(\frac{x}{a + bx}\right)$, prove that $x^3\frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2$

6. If $y = \sin^{-1} x$, then show that $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$

7. If $e^y (x + 1) = 1$, show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$

8. If $y = \left\{x + \sqrt{x^2 + 1}\right\}^m$, show that $(x^2 + 1) y_2 + xy_1 - m^2y = 0$.

9. If $y = \frac{\sin^{-1}}{\sqrt{1-x^2}}$, show that $(1 - x^2)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - y = 0$.

10. If $x = \tan\left(\frac{1}{a}\log y\right)$, show that $(1 + x^2)\frac{d^2y}{dx^2} + (2x - a)\frac{dy}{dx} = 0$.

11. $y = x^x$, prove that $\frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0$.

12. If $x = a \cos^3\theta$, $y = a \sin^3\theta$, find $\frac{d^2y}{dx^2}$.

13. If $x = a \cos \theta + b \sin \theta$ and $y = a \sin \theta - b \cos \theta$, prove that $y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$.

14. If $(x - a)^2 + (y - b)^2 = c^2$, prove that $\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$ is a constant independent of a and b.

15. If $f(x) = |x|^3$, show that $f''(x)$ exists for all real x and find it.

16. Find the second ordere derivatives of each of the following functions :

(i) $x^3 + \tan x$ (ii) $\sin(\log x)$ (iii) $\log(\sin x)$ (iv) $e^x \sin 5x$

(v) e^{6x} os $3x$ (vi) $x^3 \log x$ (vii) $\tan^{-1} x$ (viii) $x \cos x$

(ix) $\log(\log x)$

17. If $y = x + \tan x$, show that $\cos^2 x\frac{d^2y}{dx^2} - 2y + 2x = 0$

18. If $x = a(\cos \theta + \theta \sin \theta)$, $y = a(\sin \theta - \theta \cos \theta)$; prove that $\frac{d^2y}{dx^2} = \frac{\sec^3\theta}{a\theta}$.

19. If $x = a(1 - \cos \theta)$, $y = a(\theta + \sin \theta)$, prove that $\frac{d^2y}{dx^2} = \frac{-1}{a}$ at $\theta = \frac{\pi}{2}$.

20. If $x = \sin t$, $y = \sin pt$, prove that $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0$.

21. If $y = 3 \cos(\log x) + 4 \sin(\log x)$, prove that $x^2y_2 + xy_1 + y = 0$.

22. If $y = A e^{-kt} \cos(pt + c)$, prove that $\frac{d^2y}{dx^2} + 2k\frac{dy}{dx} + n^2y = 0$, where $n^2 = p^2 + k^2$.

23. If $x = \sin\left(\frac{1}{a}\log y\right)$, show that $(1 - x^2)\, y_2 - xy_1 - a^2y = 0$.

24. If $y = (\tan^{-1} x)^2$, then prove that $(1 + x)^2)^2\, y_2 + 2x(1 + x^2)y_1 = 2$.

25. Find $\frac{d^2y}{dx^2}$, where $y = \log\left(\frac{x^2}{e^2}\right)$.

26. If $y = \cos^{-1} x$, find $\frac{d^2y}{dx^2}$ in terms of y alone.

27. If $y = 500\, e^{7x} + 600\, e^{-7x}$, show that $\frac{d^2y}{dx^2} = 49y$

28. If $x = 2 \cos t - \cos 2t$, $y = 2 \sin t - \sin 2t$, find $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{2}$

29. If $y = \sin(\log x)$, prove that $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

30. If $y = 3\, e^{2x} + 2\, e^{3x}$, prove that $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

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INCREASING & DECREASING FUNCTIONS

& Their Properties

CONTENTS

THINGS TO REMEMBER

1. A function f(x) is said to be a strictly increasing function on (a, b) if

   $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_1 \in (a, b)$

   If $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_1 \in (a, b)$

   then f(x) is said to be strictly decreasing on (a, b).

2. A function f(x) is said to be monotonic on (a, b) if it is either strictly increasing or strictly decreasing on (a, b).

3. A function f(x) is said to be increasing (decreasing) at a point x_0 it there is an interval $(x_0 - h, x_0 + h)$ containing x_0 such that f(x) is increasing (decreasing on $(x_0 - h, x_0 + h)$.

4. A function f(x) is said to be increasing on [a, b] if it is increasing (decreasing) on (a, b) and it is increasing (decreasing) at x = a and x = b.

5. The necessaryand sufficient condition for a differentiable function defined on (a, b) and it is increasing (decreasing on (a, b) is that $f'(x) > 0$ for all $x \in (a, b)$.

6. The necessary and sufficient condition for a differentiable function defined on (a, b) to be strictly decreasing on (a, b) is that $f'(x) < 0$ for all $x \in (a, b)$.

7. Let f(x) be a function defined on (a, b).

   (a) If $f'(x) > 0$ for all $x \in (a, b)$ except for a finite number of points, where $f'(x) = 0$, then f(x) is increasing on (a, b).

   (b) If $f'(x) < 0$ for all $x \in (a, b)$ except for a finite number of points, where $f'(x) = 0$, then f(x) is decreasing on (a, b).

8. (i) If f(x) is strictly increasing function on a interval [a, b], then f^{-1} exists and it is also a strictly increasing function.

   (ii) If f(x) is strictly increasing function on an interval [a, b] such that it is continuous, then f^{-1} is continuous on [f(a), f(b)].

   (iii) If f(x) is continuous on [a, b] such that $f'(c) \geq 0$ ($f'(c) > 0$) for each $c \in (a, b)$, then f(x) is monotinically (strictly) increasing function on [a, b].

   (iv) If f(x) is continuous on [a, b] such that $f'(c) \leq 0$ ($f'(c) < 0$) for each $c \in (a, b)$, then f(x) is monotinically (strictly) increasing function on [a, b].

   (v) If f(x) and g(x) are monotonically (or strictly) increasing (or decreasing) functions on [a, b], then gof(x) is a monotonically (strictly) increasing function on [a, b].

   (vi) If one of the two functions f(x) and g(x) is strictly (or monotonically) increasing and other a strictly (monotonically) decreasing, then gof(x) is strictly (monotonically) decreasing on [a, b].

EXERCISE-1

1. Show that the function $f(x) = -3x + 12$ is strictly decreasing function on R.
2. Show that the function $f(x) = x^2$ is a strictly decreasing function on $(\infty -, 0]$
3. Show that the function $f(x) = x^2$ is neither strictly increasing nor strictly decreasing on R.
4. Prove that the function $f(x) = \log_e x$ is increasing on $(0, \infty)$.
5. Prove that $f(x) = ax + b$, where a, b are constants and $a < 0$ is a decreasing function on R.

6. Show that $f(x) = \frac{1}{1+x^2}$ is neither increasing nor decreasing on R.

7. (Sufficient Condition) Let be a differentiable real function defined on an open interval (a, b).
   (i) if $f'(x) > 0$ for all $x \in (a, b)$, then f(x) is increasing on (a, b).
   (ii) if $f'(x) > 0$ for all $x \in (a, b)$, then f(x) is decreasing on (a, b).

8. Find the intervals in which $f(x) = (x + 1)^3 (x - 3)^3$ is increasing or decreasing.

9. Find the intervals in which the function $f(x) = \log(1 + x) - \frac{2x}{2+x}$ is increasing or decreasing.

10. Find the intervals in which $f(x) = \frac{4x^2+1}{x}$ is increasing or decreasing.

11. Find the intervals in which the function given by

$$f(x) = x^2 + \frac{1}{x^3}, x \neq 0 \text{ is}$$

   (i) increasing (ii) decreasing

12. For which values of x, the function $f(x) = \frac{x}{x^2+1}$ is increasing and for which values of x, it is decreasing.

13. Separate $\left[0, \frac{\pi}{2}\right]$ into subintervals in which $f(x) = \sin 3x$ is increasing or decreasing.

14. Find the intervals in which the function f given by $f(x) = \sin x + \cos x$, $0 \leq x \leq 2\pi$ is increasing or decreasing.

15. Find the intervals in which the function of given by

$$f(x) = \frac{4\sin x - 2x - x\cos x}{2+\cos x}, 0 \leq x \leq 2\pi$$

   is (i) increasing (ii) decreasing.

16. Determine the values of x for which $f(x) = x^x$, $x > 0$ is increasing or decreasing.

17. Find the intervals in which $f(x) = \frac{x}{\log x}$ is increasing or decreasing.

18. Find the intervals in which $f(x) = 2\log(x - 2) - x^2 + 4x + 1$ is increasing or decreasing.

19. Separate the interval $\left[0, \frac{\pi}{2}\right]$ into sub–intervals in which $f(x) = \sin^4 x + \cos^4 x$ is increasing or decreasing.

20. Prove that the function $f(x) = x^3 - 3x^2 + 3x - 100$ is increasing on R.

21. Let I be an interval disjoined from (–1, 1). Prove that the function $f(x) = x + \frac{1}{x}$

22. Prove that $f(\theta) = \frac{4\sin\theta}{2\cos\theta} - \theta$ is an increasing function of θ in $\left[0, \frac{\pi}{2}\right]$.

23. Prove that the function $f(x) = x^2 - x + 1$ is neither increasing nor decreasing on $(-1, 1)$.

24. Find the least value of a such that the function $x^2 + ax + 1$ is increasing on $[1, 2]$.

25. On which of the following intervals is the function $f(x) = x^{100} + \sin x - 1$ increasing ?

(i) $\left(0, \frac{\pi}{2}\right)$ (ii) $\left(\frac{\pi}{2}, \pi\right)$ (iii) $(0, .1)$ (iv) $-1, 1)$

26. Which of the following functions are decreasing on $\left(0, \frac{\pi}{2}\right)$?

(i) cos x (ii) cos 2x (iii) tan x (iv) cos 3x

27. Find the intervals in which the following functions are increasing or decreasing.
(i) $f(x) = 10 - 6x - 2x^2$
(ii) $f(x) = x^2 + 2x - 5$
(iii) $f(x) = 6 - 9x - x^2$
(iv) $f(x) = ex^3 - 12x^2 + 18x + 15$
(v) $f(x) = 5 + 36x + 3x^2 - 2x^3$
(vi) $f(x) = x^3 - 6x^2 - 36x + 2$
(vii) $f(x) = -2x^3 - 9x^2 - 12x + 1$
(viii) $f(x)\ x^3 - 12^2 + 36x + 17$

28. Show that $f(x) = \log \sin x$ is increasing on $\left(0, \frac{\pi}{2}\right)$ and decreasing on $\left(\frac{\pi}{2}, \pi\right)$.

29. Prove that $f(x) = x^9 + 4x^7 + 11$ is an increasing function for all $x \in R$.

30. Find the intervals in which $f(x) = \log(1 + x) - \frac{x}{1+x}$ is increasing or decreasing.

31. Prove that the function $f(x) = \cos x$ is :
(i) strictly decreasing $(0, \pi)$
(ii) strictly increasing in $(\pi, 2\pi)$
(iii) neither increasing nor decreasing in $(0, 2\pi)$